DETERMINANTS AND THEIR USE IN ECONOMICS

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Abstrakt
Cramerovo pravidlo poskytuje jako jedna z metod řešení soustav lineárních rovnic výpočet řešení s použitím determinantů. Hesián můžeme použít pro zjištění, zda je daný kritický bod (podezřelý z extrému) lokální minimum nebo lokální maximum dané funkce dvou nebo více reálných proměnných. V tomto článku je použití Cramerova pravidla a Hesiánu demonstrováno na optimalizačních úlohách z ekonomické oblasti.

Abstract
The Cramer's rule provides a method of solving a system of linear equations through the use of determinants. The Hessian can be used for a test whether a given critical point is a local minimum or maximum of a function of two or more real variables. In the article the use of the Cramer's rule and the Hessian is demonstrated on economic optimization problems.

Introduction

Determinants and Cramer's rule are important tools for solving many problems in business and economy. Especially for searching an optimal solution of the maximization profit or minimization cost problems it can be very often apply. The article presents a popular introduction in this mathematical theory; the methods are demonstrated on numerical examples.

The article is organized as follows: in Section 1 is uses of Cramer's rule, in Section 2 The Hessian and in Section 3 is the Hessian use in optimization problems.

1. Uses of Cramer's rule

Cramer's rule (see [2], page 32) provides a simplified method of solving a system of n linear equations with n variables in the form of $\boldsymbol{Ax} = \boldsymbol{b}$, where $\boldsymbol{A}$ is the matrix of the coefficient, $\bar{x}$ is the vector unknowns and $\bar{b}$ is the vector constants in the right side. Cramer's rule states $\bar{x}_i = \dfrac{\det A_i}{\det A}$, where x_i is the ith unknown variable in a series of equations, $\det A$ is the determinant of the coefficient matrix, and $\det A_i$ is the determinant of a special matrix formed from the original coefficient matrix by replacing the column of coefficients of x_i with the column vector $\bar{b}$. The matrix A must be regular ($\det A \neq 0$).

Example 1.

The equilibrium conditions for two related markets (the price of pork is P_p and the price of beef is P_b) are given by

$$\begin{aligned} 18\ P_b - \quad P_p &= 87 \\ -2P_b + 36P_p &= 98 \end{aligned}.$$

Find the equilibrium price for each market ($\overline{P_p}$, $\overline{P_b}$).

A solution:

The following systems of linear equations in the matrix form:

$$\begin{bmatrix} 18 & -1 \\ -2 & 36 \end{bmatrix} \begin{bmatrix} P_b \\ P_p \end{bmatrix} = \begin{bmatrix} 87 \\ 98 \end{bmatrix}$$

Find the equilibrium price for each market ($\overline{P_p}$, $\overline{P_b}$).

$$\mathbf{A} = \begin{bmatrix} 18 & -1 \\ -2 & 36 \end{bmatrix}$$

where $\det A = 18(36) - (-1)(-2) = 646$.

$$\mathbf{A}_1 = \begin{bmatrix} 87 & -1 \\ 98 & 36 \end{bmatrix}$$

where $\det A_1 = 87(36) - (-1)98 = 3230$.

$$\mathbf{A}_2 = \begin{bmatrix} 18 & 87 \\ -2 & 98 \end{bmatrix}$$

where $\det A_2 = 18(98) - (-2)(87) = 1938$.

$$\overline{P_b} = \frac{\det A_1}{\det A} = \frac{3230}{646} = 5 \quad \text{and} \quad \overline{P_p} = \frac{\det A_2}{\det A} = \frac{1938}{646} = 3 \ .$$

The equilibrium price for each market ($\overline{P_p}$, $\overline{P_b}$)=(3,5).

Example 2.

The *IS* and *LM* equations can be reduced to the form

$$\begin{aligned} 0{,}4Y + 150\ i &= 209 \\ 0{,}1Y - 250\ i &= \ 35 \end{aligned}.$$

Find the equilibrium level of income $\overline{Y}$ and rate of interest $\bar{i}$.

A solution:

This problem is treated with Cramer's rule

$$\mathbf{A} = \begin{bmatrix} 0{,}4 & 150 \\ 0{,}1 & -250 \end{bmatrix}$$

where $\det A = 0{,}4(-250) - 150(0{,}1) = -115$.

$$\mathbf{A}_1 = \begin{bmatrix} 209 & 150 \\ 35 & -250 \end{bmatrix}$$

where $\det A_1 = 209(-250) - 150(35) = -57500.$

$$\mathbf{A}_2 = \begin{bmatrix} 0{,}4 & 209 \\ 0{,}1 & 35 \end{bmatrix}$$

where $\det A_2 = 0{,}4(35) - 209(0{,}1) = -6{,}9$.

$$\bar{Y} = \frac{\det A_1}{\det A} = \frac{-57500}{-115} = 500 \quad \text{and} \quad \bar{i} = \frac{\det A_2}{\det A} = \frac{-6{,}9}{-115} = 0{,}06 .$$

The equilibrium level of income $\bar{Y} = 500$ and rate of interest $\bar{i} = 0{,}06$.

2. The Hessian

Let $z = z(x, y)$ be a real function of two real variables. Let the the first-order derivatives $z_x = z_y = 0$ in a critical point (x_0, y_0) and let the second partial derivatives z_{xx}, z_{xy} and z_{yy} exist in a neighbourhood of a critical point $C(x_0, y_0)$.

A sufficient condition for a multivariable function $z = z(x, y)$ to be at a local optimum (where we find) is (see [3] pages 147-151)

1) $z_{xx}, z_{yy} > 0$ for a minimum

$z_{xx}, z_{yy} < 0$ for a maximum

2) $z_{xx} \cdot z_{yy} > (z_{xy})^2$.

A convenient test for this second-order condition is the Hessian. The *Hessian* |H| is a determinant composed of all the second-order partial derivatives, with the second-order direct partials on the principal diagonal and the second-order cross partials off the principal diagonal. Thus,

$$|H| = \begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix}$$

where $z_{xy} = z_{yx}$.

If the first element on the principal diagonal, the *first principal minor*, $|H_1| = z_{xx}$ is positive and the *second principal minor*

$$|H_2| = \begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix} = z_{xx} \cdot z_{yy} - (z_{xy})^2 > 0$$

the second-order conditions for a minimum are met. When $|H_1| > 0$ and $|H_2| > 0$, the Hessian |H| is called *positive definite*. A positive definite Hessian fulfills the second-order conditions for a minimum.

If the first principal minor $|H_1| = z_{xx} < 0$ and the second principal minor

$$|H_2| = \begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix} > 0$$

the second-order conditions for a maximum are met. When $|H_1| < 0$ and $|H_2| > 0$, the Hessian |H| is called *negative definite*. A negative definite Hessian fulfills the second-order conditions for a maximum.

If $|H_2| < 0$, a critical point is a saddle and if $|H_2| = 0$, the test is inconclusive.

3. Higher-order Hessians

Given $y = y(x_1, x_2, x_3)$, the third-order Hessian is

$$|H| = \begin{vmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{vmatrix}$$

where the elements are the various second-order partial derivatives of y:

$$y_{11} = \frac{\partial^2 y}{\partial x_1^2} \qquad y_{12} = \frac{\partial^2 y}{\partial x_1 \partial x_2} \qquad y_{23} = \frac{\partial^2 y}{\partial x_2 \partial x_3} \qquad \text{etc.}$$

Conditions for a local minimum or maximum depend on the signs of the first, second and third principal minors, respectively. If $|H_1| = y_{11} > 0$,

$$|H_2| = \begin{vmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{vmatrix} > 0 \qquad \text{and} \qquad |H_3| = |H| > 0$$

where $|H_3|$ is the *third principal minor*, |H| is positive definite and fulfills the second-order conditions for a minimum. If $|H_1| = y_{11} < 0$,

$$|H_2| = \begin{vmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{vmatrix} > 0 \qquad \text{and} \qquad |H_3| = |H| < 0$$

|H| is negative definite and will fulfill the second-order conditions for a maximum. Higher-order Hessians follow in analogous fashion. If all the principal minors of |H| are positive, |H| is positive definite and the second-order conditions for a relative minimum are met. If all the principal minors of |H| alternate in sign between negative and positive, |H| is negative definite and the second-order conditions for a relative maximum are met.

4. In this section the Hessian is used in optimization problems

Example 3.

Optimize the following function:

$$y = 3x_1^2 - 5x_1 - x_1x_2 + 6x_2^2 - 4x_2 + 2x_2x_3 + 4x_3^2 + 2x_3 - 3x_1x_3$$

a) The first-order conditions are

$$y_1 = \frac{\partial y}{\partial x_1} = 6x_1 - 5 - x_2 - 3x_3 = 0$$
$$y_2 = \frac{\partial y}{\partial x_2} = -x_1 + 12x_2 - 4 + 2x_3 = 0 \qquad (1)$$
$$y_3 = \frac{\partial y}{\partial x_3} = 2x_2 + 8x_3 + 2 - 3x_1 = 0$$

which in matrix form is

$$A \cdot \vec{x} = \begin{bmatrix} 6 & -1 & -3 \\ -1 & 12 & 2 \\ -3 & 2 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ -2 \end{bmatrix},$$

$$\text{where } A = \begin{bmatrix} 6 & -1 & -3 \\ -1 & 12 & 2 \\ -3 & 2 & 8 \end{bmatrix}.$$

Using Cramer's rule:

$$|A| = 6\ (92) + 1\ (-2) - 3\ (34) = 448$$
$$|A_1| = 5(92) + 1(36) - 3(32) = 400$$
$$|A_2| = 6(36) - 5(-2) - 3(14) = 184$$
$$|A_3| = 6(-32) + 1(14) - 3(34) = -8$$

Thus, $\bar{x}_1 = \frac{400}{448} \approx 0,89 \qquad \bar{x}_2 = \frac{184}{448} \approx 0,41 \qquad \bar{x}_3 = \frac{-8}{448} \approx -0,02$.

There is a critical point C(0,89; 0,41; -0,02)

b) Testing the second-order condition by taking the second-order partials of (1) to form the Hessian,

$$\begin{matrix} y_{11} = 6 & y_{12} = -1 & y_{13} = -3 \\ y_{21} = -1 & y_{22} = 12 & y_{23} = 2 \\ y_{31} = -3 & y_{32} = 2 & y_{33} = 8 \end{matrix}$$

Thus, $$|H| = \begin{vmatrix} 6 & -1 & -3 \\ -1 & 12 & 2 \\ -3 & 2 & 8 \end{vmatrix}$$

where $|H_1| = 6 > 0$ $\quad |H_2| = \begin{vmatrix} 6 & -1 \\ -1 & 12 \end{vmatrix} = 71 > 0$

and $|H_3| = |H| = |A| = 448 > 0$. With |H| positive definite.

There is a local minimum in a critical point C.
Y is minimized at the values.

Example 4.

A firm produces two goods (Q_1, Q_2) in pure competition and has the following total revenue and total cost functions:

$$\mathbf{TR} = 15Q_1 + 18Q_2 \qquad \mathbf{TC} = 2Q_1^2 + 2Q_1Q_2 + 3Q_2^2$$

The two goods are *technically related in production*, since the marginal cost of one is dependent on the level of output of the other (for example, $\frac{\partial TC}{\partial Q_1} = 4Q_1 + 2Q_2$) (see [4], [5], [6]).
Maximize profits π for the firm, (a) Cramer's rule for the first-order condition and (b) the Hessian for the second-order condition.

(a) $$\pi = TR - TC = 15Q_1 + 18Q_2 - 2Q_1^2 - 2Q_1Q_2 - 3Q_2^2$$

The first-order conditions are

$$\frac{\partial \pi}{\partial Q_1} = \pi_1 = 15 - 4Q_1 - 2Q_2 = 0$$
$$\frac{\partial \pi}{\partial Q_2} = \pi_2 = 18 - 2Q_1 - 6Q_2 = 0$$

In matrix form,

$$A \cdot \vec{Q} = \begin{bmatrix} -4 & -2 \\ -2 & -6 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} -15 \\ -18 \end{bmatrix},$$
$$\text{where } A = \begin{bmatrix} -4 & -2 \\ -2 & -6 \end{bmatrix}.$$

Solving by Cramer's rule,

$$|A| = 24 - 4 = 20 \qquad |A_1| = 90 - 36 = 54 \qquad |A_2| = 72 - 30 = 42$$

Thus, $$\overline{Q}_1 = \frac{54}{20} = 2{,}7 \qquad \overline{Q}_2 = \frac{42}{20} = 2{,}1$$

There is a critical point C(2,7; 2,1).

(b) Using the Hessian to test for second-order condition,

$$|H| = \begin{vmatrix} -4 & -2 \\ -2 & -6 \end{vmatrix}$$

where $|H_1| = -4$ and $|H_2| = 20$. Hessian is negative definite.
There is a local maximum in a critical point C. Profits π is maximized.

Example 5.

Maximize profits for a producer of two substitute goods, given

$$P_1 = 130 - 4Q_1 - Q_2 \qquad P_2 = 160 - 2Q_1 - 5Q_2 \qquad \mathbf{TC} = 2Q_1^2 + 2Q_1Q_2 + 4Q_2^2$$

Use (a) Cramer's rule for the first-order condition and (b) the Hessian for the second-order condition.

a) $\pi = TR - TC$, (see [4], [6]), where $\mathbf{TR} = P_1Q_1 + P_2Q_2$.

$$\pi = (80 - 5Q_1 - 2Q_2)Q_1 + (50 - Q_1 - 3Q_2)Q_2 - (2Q_1^2 + 2Q_1Q_2 + 4Q_2^2)$$

$$= 130Q_1 + 160Q_2 - 5Q_1Q_2 - 6Q_1^2 - 9Q_2^2$$

$$\frac{\partial \pi}{\partial Q_1} = \pi_1 = 130 - 5Q_2 - 12Q_1 = 0 \qquad \frac{\partial \pi}{\partial Q_2} = \pi_2 = 160 - 5Q_1 - 18Q_2 = 0$$

In matrix form,

$$A \cdot \vec{Q} = \begin{bmatrix} -12 & -5 \\ -5 & -18 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} -130 \\ -160 \end{bmatrix},$$

where $A = \begin{bmatrix} -12 & -5 \\ -5 & -18 \end{bmatrix}$.

$$|A| = 191$$

$$|A_1| = 1540 \qquad \overline{Q}_1 = \frac{1540}{191} \approx 8{,}06$$

$$|A_2| = 1270 \qquad \overline{Q}_2 = \frac{1270}{191} \approx 6{,}65$$

There is a critical point C (8,06; 6,65).

b)

$$|H| = \begin{vmatrix} -12 & -5 \\ -5 & -18 \end{vmatrix}$$

where $|H_1| = -12$ and $|H_2| = 191$. Hessian is negative definite.

There is a local maximum in a critical point C. Profits π is maximized.

Conclusion

The equilibrium of the markets in IS-LM model is solved by using determinants and Cramer's rule for a system of linear equations. Hessian matrix for test of optimal solution is presented in the problem cost minimization and profit maximization in a company.

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