SIMPLE ECONOMIC APPLICATIONS OF MATRICES

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Abstract: This article deals with simple examples pointing out to the use of mathematical models, especially of linear algebra tools (matrix operations, eigenvectors, Markov chains, systems of linear equations, least squares approximation), in economic applications. In some cases, solved problems and data given in examples may have been simplified compared to real values as the main aim is to show possibilities of applications rather than to obtain accurate results. The presented examples can help to introduce students of economics-oriented universities into the matrix theory and illustrate the connection between theory and practice.

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Introduction

In the process of education it is often advisable to show the connection between theory and practice. In this article several economic applications of matrices are presented – from quite simple to more complicated. These examples are simple so that difficult economic theories do not have to be explained. While the solved problems and data given in examples can be simplified in comparison with real values for easier computation, the economic significance is not affected. It is important to show possibilities of applications rather than to obtain an accurate result. On the other hand, it is interesting to include some real problems if possible.

1 Matrix operations

1.1 Scalar multiplication

Formulation of the problem

A store discounts commodities c_1, c_2, c_3, c_4 by 30 percent at the end of the year. The values of stocks in its three branches B_1, B_2, B_3 prior to the discount are given in Table 1. Using matrices, find the value of stock in each of B_1, B_2, B_3 after the discount.

Table 1: Values of stocks in B_1, B_2, B_3

	c_1	c_2	c_3	c_4
B_1	65000	40000	55000	35000
B_2	50000	30000	60000	45000
B_3	70000	55000	75000	50000

Source: Illustrative data

Solution

A 30 percent reduction means that the commodities are being sold for 70 percent of their original value. If we organize the information given in Table 1 into a matrix V_1, then the matrix

$$V_2 = 0.7\, V_1$$

expresses the value of stock after the discount:

$$V_2 = 0.7\ V_1 = 0.7\begin{pmatrix} 65000 & 40000 & 55000 & 35000 \\ 50000 & 30000 & 60000 & 45000 \\ 70000 & 55000 & 75000 & 50000 \end{pmatrix} = \begin{pmatrix} 45500 & 28000 & 38500 & 24500 \\ 35000 & 21000 & 42000 & 31500 \\ 49000 & 38500 & 52500 & 35000 \end{pmatrix}.$$

1.2 Distributive law for matrix multiplication

Formulation of the problem

A store sells commodities c_1, c_2, c_3 in two its branches B_1, B_2. The quantities of the commodities sold in B_1, B_2 in a week are given in Table 2, the individual prices of the commodities are given in Table 3, the costs to the store are given in Table 4. Find the store's profit for a week, using
a) total concepts
b) per-unit analysis
to show that matrix multiplication is distributive.

Table 2: Quantities of commodities

	c_1	c_2	c_3
B_1	200	350	100
B_2	250	400	150

Source: (Table 2, 3, 4) Illustrative data

Table 3: Selling prices

c_1	2.00
c_2	4.00
c_3	5.00

Table 4: Costs

c_1	1.50
c_2	3.00
c_3	4.00

Solution

The quantities (Q) of the commodities, the selling prices (P) and the costs (C) of the commodities can be represented in matrix form:

$$Q = \begin{pmatrix} 200 & 350 & 100 \\ 250 & 400 & 150 \end{pmatrix} \qquad P = \begin{pmatrix} 2.00 \\ 4.00 \\ 5.00 \end{pmatrix} \qquad C = \begin{pmatrix} 1.50 \\ 3.00 \\ 4.00 \end{pmatrix}.$$

a) Using total concepts:
The total revenue in B_1 is:

$$200 \cdot 2.00 + 350 \cdot 4.00 + 100 \cdot 5.00 = 2300$$

and in B_2:

$$250 \cdot 2.00 + 400 \cdot 4.00 + 150 \cdot 5.00 = 2850.$$

These calculations can be written using a product of two matrices: the total revenue (TR) is given by the matrix QP:

$$TR = QP = \begin{pmatrix} 200 & 350 & 100 \\ 250 & 400 & 150 \end{pmatrix}\begin{pmatrix} 2.00 \\ 4.00 \\ 5.00 \end{pmatrix} = \begin{pmatrix} 2300 \\ 2850 \end{pmatrix}.$$

Similarly, the total cost (TC) is given by the matrix QC:

$$TC = QC = \begin{pmatrix} 200 & 350 & 100 \\ 250 & 400 & 150 \end{pmatrix}\begin{pmatrix} 1.50 \\ 3.00 \\ 4.00 \end{pmatrix} = \begin{pmatrix} 1750 \\ 2175 \end{pmatrix}.$$

Profits (Π) are

$$\Pi = TR - TC = QP - QC = \begin{pmatrix} 2300 \\ 2850 \end{pmatrix} - \begin{pmatrix} 1750 \\ 2175 \end{pmatrix} = \begin{pmatrix} 550 \\ 675 \end{pmatrix}.$$

b) Using per-unit analysis:
The per-unit profit (U) is

$$U = P - C = \begin{pmatrix} 2.00 \\ 4.00 \\ 5.00 \end{pmatrix} - \begin{pmatrix} 1.50 \\ 3.00 \\ 4.00 \end{pmatrix} = \begin{pmatrix} 0.50 \\ 1.00 \\ 1.00 \end{pmatrix}.$$

The total profit (Π) is given by the matrix QU:

$$\Pi = QU = \begin{pmatrix} 200 & 350 & 100 \\ 250 & 400 & 150 \end{pmatrix} \begin{pmatrix} 0.50 \\ 1.00 \\ 1.00 \end{pmatrix} = \begin{pmatrix} 550 \\ 675 \end{pmatrix}.$$

From a) and b) ve have $\Pi = QP - QC$ and $\Pi = QU = Q(P - C)$. Thus

$$QP - QC = Q(P - C).$$

1.3 Product of matrices

Formulation of the problem

Let us consider five factories, each of them needs to supply each other. Determine the number of possibilities to transport commodities from one factory to the other one with one change at most, if there are truck and train connections as given thereinafter.

Solution

Let us denote the factories as 1, 2, 3, 4, 5 and define the 5×5 matrix A: $a_{ii} = 0$ and for $i \neq j$ we put $a_{ij} = 1$ if there is a truck connection between i and j, and $a_{ij} = 0$ otherwise. Because of two way connection between i and j the matrix A is symmetric.

For example the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

expresses that there is a truck connection between factory 1 and factory 2 (as $a_{12} = a_{21} = 1$), whereas there is no connection between factory 1 and factory 3 (as $a_{13} = a_{31} = 0$) etc.

In a similar way we define the matrix T representing a train connection:

$$T = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

Then the matrix

$$AT = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 2 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 2 & 1 & 2 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

gives the number of possibilities how to get from one of the factories to the other one in two steps – by truck at first and then by train. If we denote the elements of the matrix AT as $(at)_{ij}$, we can see that for example $(at)_{45} = 2$ – there are two possibilities how to get from the

factory 4 to the factory 5 (but not from the factory 5 to the factory 4 – the order train-truck makes a difference).
For example, since

$$(at)_{45} = a_{41}\, t_{14} + a_{42}\, t_{25} + a_{43}\, t_{35} + a_{44}\, t_{45} + a_{45}\, t_{55} = 0+1+1+0+0,$$

it is possible to get from factory 4 to factory 5 changing in 2 or in 3.
Similarly, the matrix

$$TA = \begin{pmatrix} 1 & 2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 2 & 0 \end{pmatrix}$$

gives the number of possibilities how to get from one of the factories to another one by a train at first and then by a truck.
While there are for example two possibilities of connection between factory 4 and factory 5 in the order truck-train, there is no connection between 4 and 5 in the order train-truck.
The number of all the connections between the particular factories with one change at most is given by the matrix

$$A + T + AT + TA = \begin{pmatrix} 2 & 3 & 2 & 3 & 2 \\ 3 & 0 & 2 & 3 & 2 \\ 2 & 2 & 0 & 3 & 2 \\ 3 & 3 & 3 & 2 & 3 \\ 2 & 2 & 2 & 3 & 0 \end{pmatrix}.$$

For example, we can see that there are three possibilities to get from the factory 2 to the factory 4 – one possibility of a direct connection (as a_{24}=1) and two possibilities with a change – one in the order truck-train ($(at)_{24}$ =1) and one in the order train-truck ($(ta)_{24}$ =1).

2 System of linear equations

Formulation of the problem

Average salaries in Czech Republic for the years 1990, 1995, 2000 and 2005 are subsequently 3 286, 8172, 13 219, 18 344 crowns. The values, rounded in order to compute easily without a calculator, are given in Table 5.

Table 5: Average salaries in Czech Republic

	1990	1995	2000	2005
Thousands of crowns	3	8	13	18

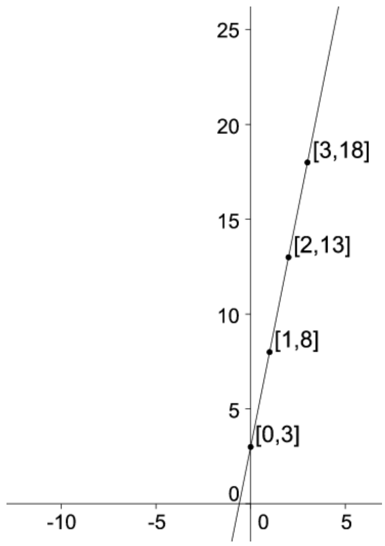
Source: Web portal Kurzycz [online]. Available: http://www.kurzy.cz/makroekonomika/mzdy/

Taking into consideration these data, we estimate the average salary in 2015.

Solution

We consider the time to be an independent variable and the salary a dependent variable – we obtain a function of one variable. If we denote the year 1990 as 0, 1995 as 1, 2000 as 2 and 2005 as 3, then we have four points [0,3], [1,8], [2,13], [3,18] in plane. We are looking for a function whose graph approximately goes through these points. If we draw the picture, we can see that the points seem to lie on a straight line (see Figure 1).

Figure 1: A line passing through the given points



Source: author's own processing

We will try to find a linear function so that coordinates of the given points satisfy the equation $y = ax+b$ of this function:

$$\begin{array}{rcrcr} a\cdot 0 & + & b & = & 3 \\ a\cdot 1 & + & b & = & 8 \\ a\cdot 2 & + & b & = & 13 \\ a\cdot 3 & + & b & = & 18. \end{array}$$

We obtained a system of linear equations represented by the augmented matrix

$$\left(\begin{array}{cc|c} 0 & 1 & 3 \\ 1 & 1 & 8 \\ 2 & 1 & 13 \\ 3 & 1 & 18 \end{array}\right) \approx \left(\begin{array}{cc|c} 1 & 1 & 8 \\ 0 & 1 & 3 \end{array}\right).$$

The solution is $a = 5$, $b = 3$. It is really surprising that the linear system has a solution – the given points lie exactly on a straight line! The equation of this line is

$$y = 5x+3.$$

To estimate the average salary in 2015 we substitute $x = 5$ (the value corresponding to the year 2015) into the equation $y = 5x+3$ and obtain $y = 28$.

Let us note that if the system has no solution (the given points do not lie on a straight line) it is possible to find the least squares solution of this system - see below.

3 Least squares approximation

Formulation of the problem

The ratios of households in Czech Republic having computers (in percentage) are given in Table 6.

We find the least squares approximating parabola for these data, compute the norm of the least squares error and estimate the ratio of households having computers in the year 2010.

Table 6: Ratios of households with computers

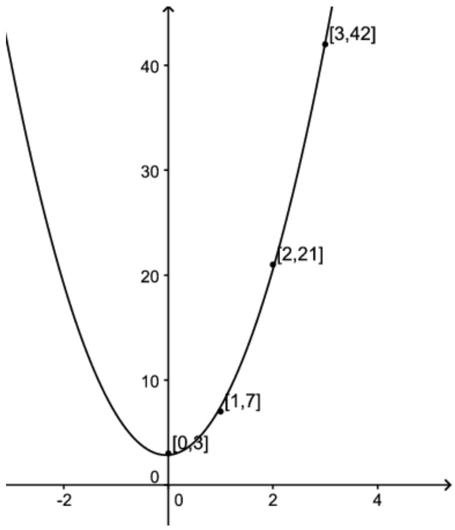
	%
1990	3
1995	7
2000	21
2005	42

Source: Web portal of Czech Statistical Office: Česká republika od roku 1989 v číslech. [online]. Available: http://www.czso.cz/cz/cr_1989_ts/0803.pdf

Solution

As in the previous problem we denote the year 1990 as 0, 1995 as 1, 2000 as 2 and 2005 as 3. In this case it is obvious that the points [0,3], [1,7], [2,21], [3,42] do not lie on a straight line (see Figure 2).

Figure 2: A least squares approximating parabola



Source: author's own processing

We will try to find a parabola that gives the least squares approximation to the given four points. Substituting these points into the equation of a parabola $y = ax^2+bx+c$, we obtain the system of linear equations

$$\begin{array}{ccccccc} a\cdot 0 & + & b\cdot 0 & + & c & = & 3 \\ a\cdot 1 & + & b\cdot 1 & + & c & = & 7 \\ a\cdot 4 & + & b\cdot 2 & + & c & = & 21 \\ a\cdot 9 & + & b\cdot 3 & + & c & = & 42 \end{array}$$

represented by the augmented matrix

$$\left(\begin{array}{ccc|c} 0 & 0 & 1 & 3 \\ 1 & 1 & 1 & 7 \\ 4 & 2 & 1 & 21 \\ 9 & 3 & 1 & 42 \end{array}\right) \approx \left(\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 0 & 2 & 3 & 7 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 3 \end{array}\right).$$

This system is inconsistent – the four given points do not lie on a parabola. To find a parabola that fits the points as much as possible, we use the least squares approximation. The least squares solution (Poole, 2003, p. 581) of the system

$$A\bar{x} = \bar{b} \quad (A_{m\times n}, \bar{x} \in R^n, \bar{b} \in R^m) \tag{1}$$

is the vector $\tilde{x}$ such that

$$\| \bar{b} - A\tilde{x} \| \le \| \bar{b} - A\bar{x} \| \text{ for all } \bar{x} \in R^n \tag{2}$$

and can be obtained as a solution of the equation

$$A^{\mathrm{T}} A \bar{x} = A^{\mathrm{T}} \bar{b} . \tag{3}$$

In our case we have

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{pmatrix}, \quad \bar{b} = \begin{pmatrix} 3 \\ 7 \\ 21 \\ 42 \end{pmatrix}, \quad \bar{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

We compute products $A^{\mathrm{T}} A$ and $A^{\mathrm{T}} \bar{b}$, put into the equation $A^{\mathrm{T}} A \bar{x} = A^{\mathrm{T}} \bar{b}$ and obtain the equation

$$\begin{pmatrix} 98 & 36 & 14 \\ 36 & 14 & 6 \\ 14 & 6 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 469 \\ 175 \\ 73 \end{pmatrix}$$

that is the system of linear equations

$$\left(\begin{array}{ccc|c} 98 & 36 & 14 & 496 \\ 36 & 14 & 6 & 175 \\ 14 & 6 & 4 & 73 \end{array} \right).$$

The solution of this system is a = 4.25, b = 0.35, c = 2.85 and

$$\tilde{x} = (4.25, 0.35, 2.85)^{\mathrm{T}}.$$

The desired equation of parabola is

$$y = 4.25x^2+0.35x+2.85.$$

To obtain the least squares error

$$\| \bar{e} \| = \| \bar{b} - A\tilde{x} \| \tag{4}$$

we compute the product $A\tilde{x}$ at first:

$$A\tilde{x} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{pmatrix} \begin{pmatrix} 4.25 \\ 0.35 \\ 2.85 \end{pmatrix} = \begin{pmatrix} 2.85 \\ 7.45 \\ 20.55 \\ 42.15 \end{pmatrix}.$$

Then

$$\| \bar{e} \| = \|(0.15, -0.45, 0.45, -0.15)^{\mathrm{T}}\| = \sqrt{0.45}\,.$$

We estimate the ratio of households having computers in the year 2010 substituting $x = 4$ into the equation $y = 4.25x^2+0.35x+2.85$: $y = 72.25 \cong 72$. Thus, using the data in the table 3 the ratio of households having computers in the year 2010 is estimated at 72%.

4 Eigenvector

Formulation of the problem

Three producers (denoted by P_1, P_2, P_3) organized in a simple closed society produce three commodities c_1, c_2, c_3. Each of these producers sells and buys from each other, all their products are consumed by them and no other commodities enter the system (the "closed model" (Friedberg, 2003, p. 176)). The proportions of the products consumed by each of P_1, P_2, P_3 are given in Table 7:

Table 7: Proportions of products c_1, c_2, c_3 consumed by P_1, P_2, P_3

	c_1	c_2	c_3
P_1	0.6	0.2	0.3
P_2	0.1	0.7	0.2
P_3	0.3	0.1	0.5

Source: Illustrative data

For example, the first column lists that 60% of the produced commodity c_1 are consumed by P_1, 10% by P_2 and 30% by P_3. As we can see, the sum of elements in each column is 1.
Let us denote x_1, x_2, x_3 the incomes of the producers P_1, P_2, P_3. Then the amount spent by P_1 on c_1, c_2, c_3 is 0.6 x_1+0.2 x_2+0.3 x_3.
The assumption that the consumption of each person equals his or her income leads to the equation 0.6 x_1+0.2 x_2+0.3 x_3 = x_1, similarly for producers P_2, P_3. We obtain the system of linear equations

$$\begin{array}{ccccccc} 0.6x_1 & + & 0.2x_1 & + & 0.3x_1 & = & x_1 \\ 0.1x_1 & + & 0.7x_1 & + & 0.2x_1 & = & x_1 \\ 0.3x_1 & + & 0.1x_1 & + & 0.5x_1 & = & x_1 \end{array}$$

This system can be rewritten as the equation $A\bar{x} = \bar{x}$, where

$$A = \begin{pmatrix} 0.6 & 0.2 & 0.3 \\ 0.1 & 0.7 & 0.2 \\ 0.3 & 0.1 & 0.5 \end{pmatrix}$$

and

$$\bar{x} = (x_1, \quad x_2, \quad x_3)^T.$$

Moreover, we assume the income to be nonnegative, i.e. $x_i \geq 0$, i=1, 2, 3 (we denote $\bar{x} \geq \bar{o}$).
We can rewrite the equation $A\bar{x} = \bar{x}$ into the equivalent form $(A–I)\bar{x} = \bar{o}$, represented by the augmented matrix

$$\left(\begin{array}{ccc|c} -0.4 & 0.2 & 0.3 & 0 \\ 0.1 & -0.3 & 0.2 & 0 \\ 0.3 & 0.1 & -0.5 & 0 \end{array}\right).$$

It means that we are looking for an eigenvector of A corresponding to the eigenvalue 1.
The general solution of the system has the form $\bar{x} = t(13,11,10)^T$; the condition $\bar{x} \geq \bar{o}$ is satisfied for $t \geq 0$.
Thus, to ensure that this society operates, the incomes of the producers P_1, P_2, P_3 have to be in the proportions 13:11:10.

5 Markov chains. Eigenvector

Formulation of the problem

Suppose a market research monitoring a group of 300 people, 200 of them use a product *A* and 100 use a product *B*. In any month 80% of product *A* users continue to use it and 20% switch to the product *B* and 90% of product *B* users continue to use it and 10% switch to the product *A*. The percentages of users loyal to the original product are assumed to be constant in next months – it means the probability of changing from one product to the other is always the same. That is a simple example of so called Markov chains. Following this research we determine, how many people will be using each product one, two and *k* months, respectively, later, and estimate the state in the long run.

Solution

The number of product *A* users after one month is given by the following formula

$$0.8\cdot 200+0.1\cdot 100=170,$$

since 80% of 200 *A* users (that is $0.8\cdot 200$) stay with *A* and in addition 10% of 100 *B* users (that is $0.1\cdot 100$) convert to *A*.

Similarly, the number of product *B* users is given by the following formula

$$0.2\cdot 200+0.9\cdot 100=130,$$

We can rewrite these two formulas using the matrix (transition matrix)

$$T=\begin{pmatrix}0.8 & 0.1\\ 0.2 & 0.9\end{pmatrix}$$

(an entry t_{ij} represents the probability of moving from state corresponding to *i* to state corresponding to *j*):

$$\begin{pmatrix}0.8 & 0.1\\ 0.2 & 0.9\end{pmatrix}\cdot\begin{pmatrix}200\\ 100\end{pmatrix}=\begin{pmatrix}170\\ 130\end{pmatrix}.$$

If we denote $\bar{x}_0 \geq \bar{o} = (200,100)^T$ (initial vector) and $\bar{x}_1 = (170,130)^T$, we can write

$$\bar{x}_1 = T\bar{x}_0 .$$

Numbers of each of *A* and *B* users after one month are given by the components of the vector $\bar{x}_1$ (let us note that these components are not necessarily integers – they are only approximations of numbers of people).

Similarly to computing numbers of users after one month we determine numbers of users after two months (represented by the vector $\bar{x}_2$):

$$\bar{x}_2=T\bar{x}_1=\begin{pmatrix}0.8 & 0.1\\ 0.2 & 0.9\end{pmatrix}\cdot\begin{pmatrix}170\\ 130\end{pmatrix}=\begin{pmatrix}149\\ 151\end{pmatrix}$$

(we can also write $\bar{x}_2 = T\bar{x}_1 = T(T\ \bar{x}_0) = T^2\bar{x}_0$).

It is obvious that numbers of *A* and *B* users after *k* months are determined by

$$\bar{x}_k=T\bar{x}_{k-1}\quad\text{or}\quad \bar{x}_k=T^k\bar{x}_0 \tag{5}$$

(an entry $(t^k)_{ij}$ of this matrix T^k represents the probability of moving from state corresponding to *i* to state corresponding to *j* in *k* transitions).

It is possible to show (Poole, 2003, p. 323) that if the transition matrix is a matrix, which has some power positive, then vectors $\bar{x}_k$ (state vectors) converge for large *k* to an unique vector $\bar{x}$ (steady state vector); when this vector is reached, it will not change by multiplying by *T*:

$$\bar{x}=T\bar{x} \tag{6}$$

(it means that *T* has 1 as an eigenvalue and the steady state vector is one of eigenvectors corresponding to this eigenvalue). Moreover, steady state vector does not depend on the choice of the initial vector $\bar{x}_0$. To determine numbers of *A* and *B* users after a long time we compute the steady state vector (*I* is a unit matrix):

$$\bar{x}=T\bar{x}\quad\Leftrightarrow\quad T\bar{x}-\bar{x}=\bar{o}\quad\Leftrightarrow\quad (T-I)\bar{x}=\bar{o}\,. \tag{7}$$

Since

$$T-I=\begin{pmatrix}-0.2 & 0.1\\ 0.2 & -0.1\end{pmatrix},$$

we obtain the following homogeneous linear system

$$\left(\begin{array}{cc|c}-0.2 & 0.1 & 0\\ 0.2 & -0.1 & 0\end{array}\right)$$

the general solution of which is

$$\bar{x} = (t, 2t)^T.$$

Since components of each of state vectors represent numbers of *A* resp. *B* users, the sum of these components must be equal to the global number of users; in our case components of the steady stage vector must satisfy

$$t+2t = 300,$$

from which it follows that t=100 and

$$\bar{x} = (100, 200)^T.$$

After a long time, 100 people will be using the product *A* and 200 people will be using the product *B* (and this result does not depend on the initial distribution of *A* and *B* users).

Conclusion

The aim of this article is to show easy examples, which point out to the use of matrices in economic applications. To use the tools of linear algebra for solving some real problems it is usually necessary to have deeper knowledge of linear algebra or other branches of science. In this article a few examples motivated by tasks in economy are provided. Their solutions illustrate the use of linear algebra tools such as matrix operations, eigenvectors, Markov chains, system of linear equations and least squares approximation. These examples can be used in mathematical courses taught at economics-oriented universities.

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